

# Scale without Conformal Invariance: Theoretical Foundations

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We present the theoretical underpinnings of scale without conformal invariance in quantum field theory. In light of our results the gradient-flow interpretation of renormalization-group (RG) flow is challenged, due to deep connections between scale-invariant theories and recurrent behaviors in the RG. We show that, on scale-invariant trajectories, there is a redefinition of the dilatation current that leads to generators of dilatations that generate dilatations. Finally, we develop a systematic algorithm for the search of scale-invariant trajectories in perturbation theory.

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## 1. Introduction

It has long been presumed that, under mild assumptions, scale invariance implies conformal invariance in relativistic quantum field theory. Although no proof is known in  $d > 2$  spacetime dimensions, until very recently [1] a credible counterexample was lacking.

In  $d = 2$  spacetime dimensions, Polchinski [2], following an argument of Zamolodchikov [3], proved that scale invariance implies conformal invariance for unitary quantum field theories with finite correlation functions. The technical assumptions of unitarity and finiteness of the correlation functions play an important role in the proof, and counterexamples which do not satisfy these assumptions have been found. Indeed, a scale-invariant model without conformal invariance, in which correlation functions of the energy-momentum tensor are not finite, was discovered by Hull and Townsend [4]. Moreover, the theory of elasticity in  $d = 2$  Euclidean dimensions, a non-reflection-positive theory, was shown by Cardy and Riva to display scale but not conformal invariance [5]. Other counterexamples are also known [6]—nevertheless unitarity or finiteness of the correlation functions is violated in each case.<sup>1</sup>

Polchinski also showed that a unitary scale-invariant theory of scalar fields in  $d = 4 - \epsilon$  is automatically conformally invariant at one-loop order [2]. The argument he used is simple: for couplings  $g_i$  with beta functions  $\beta^i$ , scale invariance implies  $\beta^i = \mathcal{Q}^i$  with  $\mathcal{Q}^i \neq 0$ , while conformal invariance requires  $\mathcal{Q}^i = 0$  (see Eqs. (2.8)). Therefore, if by direct computation one shows that  $(\mathcal{Q}^i)^* \beta^i = 0$ , then scale implies conformal invariance. Later on, his result was extended (also at one-loop order) to a theory of scalar fields and Weyl fermions, in  $d = 4 - \epsilon$ , by Dorigoni and Rychkov [8]. Recently, we showed that for a unitary quantum field theory of scalar fields exclusively scale implies conformal invariance to two loops, while in a unitary quantum field theory of Weyl fermions and no more than one real scalar field, scale invariance implies the vanishing of the beta functions (and hence conformal invariance) to all orders in the loop expansion [1].

More surprisingly, though, we showed that, in more general (unitary) theories, scale invariance does not necessarily imply conformal invariance beyond the one-loop order; the Polchinski–Dorigoni–Rychkov argument breaks down at two loops. For example, the “ $y^3\lambda$ ” two-loop term in the Yukawa beta functions leads to an obstruction to the Polchinski–Dorigoni–Rychkov argument. This term also generates an impediment to expressing the renormalization group (RG) flow as a gradient flow as shown by Wallace and Zia [9], and a deep connection between scale-invariant trajectories and recurrent behaviors is revealed.

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<sup>1</sup>For related work see also Ref. [7].

In this paper we will investigate the theoretical consequences of scale without conformal invariance for four-dimensional quantum field theories of vector fields, scalar fields and Weyl fermions, and we will discover remarkable properties of scale-invariant theories. As we will argue, scale-invariant trajectories correspond to rare RG flows, namely recurrent behaviors (i.e., limit cycles and ergodic behavior). Specific well-defined examples of scale-invariant trajectories in  $d = 4 - \epsilon$  (which are unitary, with finite correlation functions and energy bounded from below) and in  $d = 4$  (unfortunately with energy unbounded from below) will be discussed elsewhere. Here we content ourselves with reviewing the systematic expansion we used in [1] to search for recurrent flows.

The paper is organized as follows: section 2 reviews the conditions under which a theory is scale and/or conformally invariant. The conditions for scale without conformal invariance are then translated to conditions on the beta functions with the help of the new improved energy-momentum tensor. An analysis of the RG flow of scale-invariant trajectories leads to a connection with RG recurrent behaviors. Such behaviors imply that RG flows are not gradient flows, and interesting consequences, e.g., for the  $a$ -theorem, are discussed. Finally, an investigation of dilatation generators for scale-invariant theories allows one to conclude that dilatation generators do generate dilatations for scale-invariant theories, even with non-vanishing beta functions. The implications of scale without conformal invariance on correlation functions are also briefly discussed. Section 3 describes a general technique to discover scale-invariant trajectories that are not conformal in generic quantum field theories. Finally, we conclude in section 4.

## 2. Scale versus Conformal Invariance

### 2.1. Preliminaries

Let us first review under which circumstances a quantum field theory is scale or conformally invariant [2, 10, 11]. The most general form of the dilatation current is

$$\mathcal{D}^\mu(x) = x^\nu T_\nu^\mu(x) - V^\mu(x), \quad (2.1)$$

where  $T^{\mu\nu}(x)$  is any symmetric energy-momentum tensor and  $V^\mu(x)$ , the virial current,<sup>2</sup> is any local operator that does not explicitly depend on  $x^\mu$ . The former is determined by the spacetime nature of scale transformations, while the latter, an internal transformation,

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<sup>2</sup>Strictly speaking the “field virial” is a very specific current defined, e.g., in Eq. (A.14) of Ref. [11]. We are relaxing the strict interpretation of the term.

contributes to the scaling dimensions of the fields of the theory. Notice that the allowed freedom in the choice of the symmetric energy-momentum tensor is balanced by the liberty to arbitrarily select the current  $V^\mu(x)$ . Since it is finite and not renormalized, the new improved energy-momentum tensor  $\Theta^{\mu\nu}(x)$  [10] will be a particularly helpful choice of  $T^{\mu\nu}(x)$  in the following.

For any given choice of energy-momentum tensor, the dilatation current will be conserved and the theory will exhibit scale invariance if there exists a virial current such that

$$T_\mu{}^\mu(x) = \partial_\mu V^\mu(x). \quad (2.2)$$

For  $d > 2$  the theory will also feature conformal symmetry if the virial current is the sum of a conserved current,  $J^\mu(x)$ , and the divergence of a two-index symmetric local operator,  $L^{\mu\nu}(x)$ , such that

$$T_\mu{}^\mu(x) = \partial_\mu V^\mu(x) = \partial_\mu \partial_\nu L^{\mu\nu}(x). \quad (2.3)$$

This last statement is equivalent to the existence of a traceless symmetric energy-momentum tensor [2].

Therefore, for a quantum field theory to be scale but not conformally invariant, it is necessary that Eq. (2.2) is satisfied, with the additional requirement that Eq. (2.3) is not, i.e., the virial current is not the sum of a conserved current and the divergence of a two-index symmetric local operator:

$$T_\mu{}^\mu(x) = \partial_\mu V^\mu(x), \text{ where } V^\mu(x) \neq J^\mu(x) + \partial_\nu L^{\nu\mu}(x) \text{ with } \partial_\mu J^\mu(x) = 0. \quad (2.4)$$

The possible choices for the virial current are easily determined, since its spatial integral must be gauge-invariant and, in  $d$  spacetime dimensions, its scaling dimension must be  $d - 1$ . In a general  $d = 4$  renormalizable quantum field theory the use of the new improved energy-momentum tensor in Eq. (2.4) will be particularly useful in constraining the virial current.

## 2.2. Scale Invariance and the New Improved Energy-Momentum Tensor [10]

Using dimensional regularization, the most general classically scale-invariant  $d = 4 - \epsilon$  renormalizable theory involving gauge fields,  $A_\mu^A(x)$ , interacting with real scalar fields,  $\phi_a(x)$ , and Weyl fermions,  $\psi_i(x)$ , belonging to arbitrary representations of the gauge group,<sup>3</sup> is

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<sup>3</sup>Upper case indices from the beginning of the roman alphabet are gauge indices for vector fields. Lower case indices from the beginning of the roman alphabet are indices in flavor and gauge space for scalar fields, while lower case indices from the middle are indices in flavor and gauge space for Weyl spinors.

described by the Lagrangian

$$\begin{aligned} \mathcal{L} = & -\mu^{-\epsilon} Z_A \frac{1}{4g_A^2} F_{\mu\nu}^A F^{A\mu\nu} + \frac{1}{2} Z_{ab}^{\frac{1}{2}} Z_{ac}^{\frac{1}{2}} D_\mu \phi_b D^\mu \phi_c + \frac{1}{2} Z_{ij}^{\frac{1}{2}*} Z_{ik}^{\frac{1}{2}} \bar{\psi}_j i \bar{\sigma}^\mu D_\mu \psi_k - \frac{1}{2} Z_{ij}^{\frac{1}{2}*} Z_{ik}^{\frac{1}{2}} D_\mu \bar{\psi}_j i \bar{\sigma}^\mu \psi_k \\ & - \frac{1}{4!} \mu^\epsilon (\lambda Z^\lambda)_{abcd} \phi_a \phi_b \phi_c \phi_d - \frac{1}{2} \mu^{\frac{\epsilon}{2}} (y Z^y)_{a|ij} \phi_a \psi_i \psi_j - \frac{1}{2} \mu^{\frac{\epsilon}{2}} (y Z^y)^*_{a|ij} \phi_a \bar{\psi}_i \bar{\psi}_j, \end{aligned} \quad (2.5)$$

where  $\lambda_{abcd}$  is totally symmetric in its indices and  $y_{a|ij} = y_{a|ji}$ . For simplicity, gauge-field kinetic mixings are not considered. The kinetic terms are defined through

$$\begin{aligned} F_{\mu\nu}^A &= \partial_\mu A_\nu^A - \partial_\nu A_\mu^A + f^{ABC} A_\mu^B A_\nu^C, \\ D_\mu \phi_a &= \partial_\mu \phi_a + i \theta_{ab}^A A_\mu^A \phi_b, \\ D_\mu \psi_i &= \partial_\mu \psi_i + i t_{ij}^A A_\mu^A \psi_j, \end{aligned}$$

where the gauge-group generators  $\theta_{ab}^A$  and  $t_{ij}^A$  are Hermitian (the real-scalar-field generators  $\theta_{ab}^A$  are also purely imaginary and antisymmetric). By gauge invariance the couplings satisfy

$$\begin{aligned} \theta_{a'a}^A \lambda_{a'bcd} + \theta_{b'b}^A \lambda_{ab'cd} + \theta_{c'c}^A \lambda_{abc'd} + \theta_{d'd}^A \lambda_{abcd'} &= 0, \\ \theta_{a'a}^A y_{a'|ij} + t_{i'i}^A y_{a|i'j} + t_{j'j}^A y_{a|ij'} &= 0. \end{aligned}$$

The beta functions are given by vertex corrections plus wavefunction renormalizations:

$$\begin{aligned} \beta_A &= -\frac{dg_A}{dt} = \gamma_A g_A \quad (\text{no sum}), \\ \beta_{abcd} &= -\frac{d\lambda_{abcd}}{dt} = -(\lambda \gamma^\lambda)_{abcd} + \gamma_{a'a} \lambda_{a'bcd} + \gamma_{b'b} \lambda_{ab'cd} + \gamma_{c'c} \lambda_{abc'd} + \gamma_{d'd} \lambda_{abcd'}, \\ \beta_{a|ij} &= -\frac{dy_{a|ij}}{dt} = -(y \gamma^y)_{a|ij} + \gamma_{a'a} y_{a'|ij} + \gamma_{i'i} y_{a|i'j} + \gamma_{j'j} y_{a|ij'}. \end{aligned} \quad (2.6)$$

Here  $\gamma^\lambda$  and  $\gamma^y$  are the  $\lambda_{abcd}$  and  $y_{a|ij}$  vertex anomalous dimensions computed from the vertex corrections  $Z^\lambda$  and  $Z^y$  respectively, while  $\gamma_A$ ,  $\gamma_{ab}$  and  $\gamma_{ij}$  are the gauge-field, real-scalar-field and Weyl-fermion anomalous-dimension matrices computed from the wavefunction renormalizations  $Z_A$ ,  $Z_{ab}^{\frac{1}{2}}$  and  $Z_{ij}^{\frac{1}{2}}$ , respectively. The RG time is defined as  $t = \ln(\mu_0/\mu)$ .

In this perturbative setting the most general non-trivial candidate for the virial current is

$$V^\mu(x) = Q_{ab} \phi_a D^\mu \phi_b - P_{ij} \bar{\psi}_i i \bar{\sigma}^\mu \psi_j, \quad (2.7)$$

where  $Q_{ab}$  is antisymmetric and  $P_{ij}$  is anti-Hermitian, i.e.,  $Q_{ba} = -Q_{ab}$  and  $P_{ji}^* = -P_{ij}$ . By gauge invariance the unknown coefficients  $Q_{ab}$  and  $P_{ij}$  must satisfy

$$\begin{aligned} \theta_{a'a}^A Q_{a'b} + \theta_{b'b}^A Q_{ab'} &= 0, \\ -t_{ii'}^A P_{i'j} + t_{j'j}^A P_{ij'} &= 0. \end{aligned}$$

The unknown coefficients in Eq. (2.7) are to be determined by satisfying Eq. (2.4), which is greatly simplified once the new improved energy-momentum tensor is used. One reason for that is that the new improved energy-momentum tensor  $\Theta_{\mu\nu}(x)$  is finite and not renormalized [10].

From the Lagrangian (2.5) the trace of the new improved energy-momentum tensor is given by [12]

$$\begin{aligned}\Theta_{\mu}^{\mu}(x) = & \frac{\beta_A}{2g_A^3} F_{\mu\nu}^A F^{A\mu\nu} + \gamma_{aa'} D^2 \phi_a \phi_{a'} - \gamma_{i'i}^* \bar{\psi}_i i \bar{\sigma}^{\mu} D_{\mu} \psi_{i'} + \gamma_{ii'} D_{\mu} \bar{\psi}_i i \bar{\sigma}^{\mu} \psi_{i'} \\ & - \frac{1}{4!} (\beta_{abcd} - \gamma_{a'a} \lambda_{a'bcd} - \gamma_{b'b} \lambda_{ab'cd} - \gamma_{c'c} \lambda_{abc'd} - \gamma_{d'd} \lambda_{abcd'}) \phi_a \phi_b \phi_c \phi_d \\ & - \frac{1}{2} (\beta_{a|ij} - \gamma_{a'a} y_{a'|ij} - \gamma_{i'i} y_{a|i'j} - \gamma_{j'j} y_{a|ij'}) \phi_a \psi_i \psi_j + \text{h.c.} .\end{aligned}$$

Recall that a transformation is a symmetry of the theory if the infinitesimal transformation changes the Lagrangian by a total derivative *without* the use of the equations of motion (EOMs). However, the fact that a current is conserved is determined *with* the help of the EOMs [11]. Therefore, the dilatation current,  $\mathcal{D}^{\mu}(x)$ , the divergence of which is

$$\begin{aligned}\partial_{\mu} \mathcal{D}^{\mu}(x) = & \frac{\beta_A}{2g_A^3} F_{\mu\nu}^A F^{A\mu\nu} + (\gamma_{aa'} + Q_{aa'}) D^2 \phi_a \phi_{a'} - (\gamma_{i'i}^* + P_{i'i}^*) \bar{\psi}_i i \bar{\sigma}^{\mu} D_{\mu} \psi_{i'} + (\gamma_{ii'} + P_{ii'}) D_{\mu} \bar{\psi}_i i \bar{\sigma}^{\mu} \psi_{i'} \\ & - \frac{1}{4!} (\beta_{abcd} - \gamma_{a'a} \lambda_{a'bcd} - \gamma_{b'b} \lambda_{ab'cd} - \gamma_{c'c} \lambda_{abc'd} - \gamma_{d'd} \lambda_{abcd'}) \phi_a \phi_b \phi_c \phi_d \\ & - \frac{1}{2} (\beta_{a|ij} - \gamma_{a'a} y_{a'|ij} - \gamma_{i'i} y_{a|i'j} - \gamma_{j'j} y_{a|ij'}) \phi_a \psi_i \psi_j + \text{h.c.} ,\end{aligned}$$

is conserved if

$$\begin{aligned}\beta_A &= 0 , \\ \beta_{abcd} &= -Q_{a'a} \lambda_{a'bcd} - Q_{b'b} \lambda_{ab'cd} - Q_{c'c} \lambda_{abc'd} - Q_{d'd} \lambda_{abcd'} , \\ \beta_{a|ij} &= -Q_{a'a} y_{a'|ij} - P_{i'i} y_{a|i'j} - P_{j'j} y_{a|ij'} ,\end{aligned}\tag{2.8}$$

for the remaining terms vanish with the help of the EOMs.<sup>4</sup> The form of Eqs. (2.8) is unmodified to all orders. Moreover, since operators related to the EOMs are finite and not renormalized [12,13], the dilatation current, and consequently the virial current, is finite and not renormalized for scale-invariant theories. Furthermore, in  $d = 4$  spacetime dimensions the virial current has dimension exactly 3, although it is not conserved—the theory is

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<sup>4</sup>Note that, at the quantum level, the divergence of a current has three types of contributions: the usual contribution proportional to the EOMs, the contribution which corresponds to the classical violation of the symmetry, and a possible anomaly. Here the classical violation of the symmetry vanishes, while the combination of the anomaly and the virial current is proportional to the EOMs.

scale but not conformally invariant.<sup>5</sup> Notice that, in terms of the vertex corrections, the conditions for scale invariance imply

$$\begin{aligned} (\lambda\gamma^\lambda)_{abcd} &= (\gamma_{a'a} + Q_{a'a})\lambda_{a'bcd} + (\gamma_{b'b} + Q_{b'b})\lambda_{ab'cd} + (\gamma_{c'c} + Q_{c'c})\lambda_{abc'd} + (\gamma_{d'd} + Q_{d'd})\lambda_{abcd'} , \\ (y\gamma^y)_{a|ij} &= (\gamma_{a'a} + Q_{a'a})y_{a'|ij} + (\gamma_{i'i} + P_{i'i})y_{a|i'j} + (\gamma_{j'j} + P_{j'j})y_{a|ij'} . \end{aligned} \quad (2.9)$$

Thus, for scale invariance to hold, the vertex anomalous dimensions must have very specific forms. This fact will play an important role later: it will quantum-mechanically generate the scaling dimensions required in the dilatation generator.

Finally, it is easy to see from Eq. (2.4) that a theory is scale-invariant but not conformally invariant, if and only if there is a non-trivial virial current such that  $\partial_\mu V^\mu(x) \neq 0$  and Eqs. (2.8) or (2.9) are satisfied.

### 2.3. Renormalization Group Flow along Scale-invariant Trajectories

A quantum field theory with a non-trivial scale-invariant trajectory is a theory with a solution to Eq. (2.4), or equivalently Eqs. (2.8) or (2.9).<sup>6</sup> As mentioned in the introduction, the Polchinski–Dorigoni–Rychkov argument breaks down at two-loop order in a theory with enough scalars and fermions. Hence, scale invariance does not necessarily imply conformal invariance. Once a non-trivial scale-invariant solution  $(g_A, \lambda_{abcd}, y_{a|ij})$  has been found, it is easy to solve exactly the RG equations on the scale-invariant trajectory. Indeed, with the RG time defined as  $t = \ln(\mu_0/\mu)$ , the RG evolution on a scale-invariant trajectory is remarkably simple; it is

$$\begin{aligned} \bar{g}_A(t) &= g_A , \\ \bar{\lambda}_{abcd}(t) &= \hat{Z}_{a'a}(t)\hat{Z}_{b'b}(t)\hat{Z}_{c'c}(t)\hat{Z}_{d'd}(t)\lambda_{a'b'c'd'} , \\ \bar{y}_{a|ij}(t) &= \hat{Z}_{a'a}(t)\hat{Z}_{i'i}(t)\hat{Z}_{j'j}(t)y_{a'|i'j'} , \end{aligned} \quad (2.10)$$

where the  $\hat{Z}(t)$  matrices are given by

$$\begin{aligned} \hat{Z}_{aa'}(t) &= (e^{Q^t})_{aa'} , \\ \hat{Z}_{ii'}(t) &= (e^{P^t})_{ii'} . \end{aligned} \quad (2.11)$$

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<sup>5</sup>This argument is consistent with the different unitarity bounds for conformal versus scale-invariant field theories [14].

<sup>6</sup>Of course, conformal fixed points still satisfy Eqs. (2.8) and (2.9) with trivial virial current.

Notice that any other point  $(\bar{g}_A(t, g, \lambda, y), \bar{\lambda}_{abcd}(t, g, \lambda, y), \bar{y}_{a|ij}(t, g, \lambda, y))$  on the scale-invariant trajectory will satisfy Eqs. (2.8), since the couplings and, by the gauge transformations of  $Q_{ab}$  and  $P_{ij}$ , the beta functions and, also, the anomalous dimensions, transform homogeneously along the scale-invariant trajectory:

$$\begin{aligned}\bar{\beta}_{abcd}(t) &= \widehat{Z}_{a'a}(t)\widehat{Z}_{b'b}(t)\widehat{Z}_{c'c}(t)\widehat{Z}_{d'd}(t)\beta_{a'b'c'd'} , \\ \bar{\beta}_{a|ij}(t) &= \widehat{Z}_{a'a}(t)\widehat{Z}_{i'i}(t)\widehat{Z}_{j'j}(t)\beta_{a'|i'j'} , \\ \bar{\gamma}_{ab}(t) &= \widehat{Z}_{a'a}(t)\widehat{Z}_{b'b}(t)\gamma_{a'b'} , \\ \bar{\gamma}_{ij}(t) &= \widehat{Z}_{i'i}^*(t)\widehat{Z}_{j'j}(t)\gamma_{i'j'} .\end{aligned}\tag{2.12}$$

Here, unbarred parameters are evaluated at  $(g_A, \lambda_{abcd}, y_{a|ij})$ , i.e., at the scale-invariant solution. The behavior (2.12) ensures that  $Q_{ab}$  and  $P_{ij}$  are constant along the scale-invariant trajectory. Indeed, from Eqs. (2.8) and (2.10) one can see that trajectories with  $Q_{ab}$  and/or  $P_{ij}$  that are functions of RG time are not possible, for these trajectories would then intersect trajectories with constant  $Q_{ab}$  and  $P_{ij}$ . Such intersecting trajectories cannot occur in well-posed initial value problems.

It is perhaps surprising that scale-invariant trajectories are full-fledged RG trajectories and not simply points since they are always referred to as such in the literature. However, because scale without conformal invariance implies the non-vanishing of at least one of the beta functions, if a scale-invariant solution  $(g_A, \lambda_{abcd}, y_{a|ij})$  exists, complete scale-invariant trajectories must exist and are described by Eqs. (2.10) and (2.11). Note that because  $\widehat{Z}_{ab}(t)$  is orthogonal and  $\widehat{Z}_{ij}(t)$  unitary, if one of the beta functions is non-zero at some point on a scale-invariant trajectory, then at least one of the beta functions will be non-zero at any other point on the trajectory, i.e., the theory always flows.

Defining matrices  $\widetilde{Z}(t)$  as

$$\begin{aligned}\widetilde{Z}_{aa'}(t) &= (e^{-(\gamma^\phi + Q)t})_{aa'} , \\ \widetilde{Z}_{ii'}(t) &= (e^{-(\gamma^\psi + P)t})_{ii'} ,\end{aligned}\tag{2.13}$$

the vertex corrections and wavefunction renormalizations on scale-invariant trajectories are simply

$$\begin{aligned}Z_{abcd, a'b'c'd'}^\lambda &= (\widehat{Z}^{-1}\widetilde{Z}\widehat{Z})_{aa'}(\widehat{Z}^{-1}\widetilde{Z}\widehat{Z})_{bb'}(\widehat{Z}^{-1}\widetilde{Z}\widehat{Z})_{cc'}(\widehat{Z}^{-1}\widetilde{Z}\widehat{Z})_{dd'} , & Z_{aa'}^{\frac{1}{2}} &= (\widetilde{Z}\widehat{Z})_{aa'} , \\ Z_{a|ij, a'|i'j'}^y &= (\widehat{Z}^{-1}\widetilde{Z}\widehat{Z})_{aa'}(\widehat{Z}^{-1}\widetilde{Z}\widehat{Z})_{ii'}(\widehat{Z}^{-1}\widetilde{Z}\widehat{Z})_{jj'} , & Z_{ii'}^{\frac{1}{2}} &= (\widetilde{Z}\widehat{Z})_{ii'} .\end{aligned}\tag{2.14}$$

With the help of Eqs. (2.11) and (2.13), it is a straightforward computation to check that Eqs. (2.14) lead to scale-invariant trajectories as described by Eqs. (2.8).



## 2.4. Scale Invariance and Recurrent Behaviors

There is a close connection between theories with scale but without conformal invariance and recurrent RG behaviors. Indeed, since  $Q_{ab}$  is real antisymmetric and  $P_{ij}$  anti-Hermitian, they both have purely imaginary eigenvalues. In other words, the matrices (2.11) are elements of  $SO(N_S) \times U(N_F)$  which correspond to the symmetry group of the kinetic terms of the  $N_S$  real scalars and  $N_F$  Weyl fermions. Consequently, scale-invariant trajectories described by Eqs. (2.10) must be periodic or quasi-periodic. In other words, scale-invariant trajectories exhibit limit cycles or ergodicity, with oscillation frequencies determined by the eigenvalues of the virial current [15, 16].

The connection between scale invariance and recurrent behaviors can be understood intuitively. Indeed, RG evolution is related to the dilatation current (2.1), which is a combination of a spacetime scale transformation and an internal transformation of the fields. Since the internal transformation is a transformation in the compact group  $SO(N_S) \times U(N_F)$ , the field transformations must eventually rotate back to the identity or arbitrary close to the identity. Thus, since RG translations are generated by scale transformations, and scale transformations are related to internal transformations by the conserved dilatation current, RG evolution along scale-invariant trajectories that are not conformal must return to, or arbitrarily close to, the starting point. Hence, scale-invariant trajectories are periodic or quasi-periodic. Notice that the converse is also true: limit cycles and RG trajectories exhibiting ergodic behavior are scale-invariant trajectories with no conformal symmetry, since physical properties on an RG trajectory are independent of RG time. However, it is possible that scale invariance on such a trajectory is broken to a discrete subgroup.<sup>7</sup> These interesting RG behaviors have far-reaching consequences as discussed in the next section.

At this point some might be puzzled by the argument of the previous paragraph. Indeed, since RG running on a scale-invariant trajectory can be seen as a field redefinition, one might be tempted to argue that scale-invariant trajectories are really fixed points. However, the appropriate field redefinition is RG-time-dependent and consequently it generates an RG flow as advocated, e.g., in Ref. [17]. In other words, although the RG flow can be interpreted as a field redefinition, it is impossible to make all beta functions vanish on a

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<sup>7</sup>The physics lore that limit cycles and ergodicity imply perpetual oscillations in the scattering cross sections is thus correct [15]. Close to a scale-invariant trajectory, the scattering cross sections  $\sigma(s)$  in terms of the center-of-mass energy  $s$  oscillate, i.e.,  $s\sigma(s) = c(s)$  with  $c(s)$  a periodic or quasi-periodic function. The scattering cross sections obey the standard scaling law, with  $c(s)$  a constant, only for theories approaching conformal fixed points.

scale-invariant trajectory.<sup>8</sup>

Finally, note that recurrent behaviors are  $n$ -dimensional compact subspaces of coupling space where  $n = 1$  for limit cycles and  $n > 1$  for ergodic behaviors. Although a complete analysis of the behavior of RG trajectories near scale-invariant trajectories has not been undertaken, at first sight it seems like an extreme amount of fine-tuning is necessary for a theory to exhibit limit cycles (exactly as in theories that sit at a fixed point). The prospect does not appear as grim for ergodicity. Since the compact subspace is completely spanned (in infinite RG time), any UV theory defined in the compact subspace will display ergodicity. A careful analysis of RG trajectories near scale-invariant trajectories would shed light on the character (attractive, repulsive, etc.) of scale-invariant trajectories.

### 2.5. Scale-invariant Theories, Gradient Flows and the $a$ -theorem

It has long been known that recurrent behaviors such as limit cycles and ergodicity imply that RG flows are not gradient flows [9]. An RG flow is a gradient flow if the beta functions,  $\beta^i = -\frac{dg^i}{dt}$ , can be written as

$$\beta^i(g) = G^{ij}(g) \frac{\partial c(g)}{\partial g^j},$$

with  $G_{ij}(g)$  a positive-definite metric,  $G^{ik}G_{kj} = \delta^i_j$ , and  $c(g)$  a function of the couplings  $g^i$ . Along an RG trajectory, the potential  $c(g(t))$  is a monotonically decreasing function,

$$\frac{dc(g(t))}{dt} = -G_{ij}(g)\beta^i\beta^j \leq 0,$$

Clearly, scale-invariant trajectories cannot be produced by gradient flows.

The obstruction appears first at two loops, which is consistent with the literature [9,19]. For example, it was pointed out in [9] that a specific *one*-loop contribution to the quartic-coupling beta functions (schematically  $y^4$ ) and a specific *two*-loop contribution to the Yukawa beta functions (schematically  $y^3\lambda$ ) arise from the same  $\lambda y^4$  term in an appropriately constructed potential  $c(g)$ . Such an obstruction has important repercussions in the study of RG flows. Intuitively, a non-trivial RG flow is seen as an irreversible process, where the high-momentum degrees of freedom are integrated out at large distances. In other words, the number of massless degrees of freedom should always decrease along non-trivial RG flows. However, on scale-invariant trajectories, the number of massless degrees

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<sup>8</sup>Note that all exact RG flows can be obtained by RG-time-dependent field redefinitions, see e.g., [18]. Moreover, since wavefunction renormalization operators are redundant, it is necessary for scale invariance that the beta-function operators are redundant on scale-invariant trajectories—the beta functions can then be absorbed in the anomalous dimensions as discussed in section 2.6.

of freedom is constant. This implies that the “strongest” version of the  $a$ -theorem [20], i.e., that RG flows are gradient flows, is wrong. Note that the “stronger” claim—that the potential  $c(g)$  is monotonically decreasing,  $dc/dt \leq 0$ —still stands. The inequality is saturated if the theory is scale-invariant.

A supersymmetric example of scale without conformal invariance is still missing, and so it is possible that scale implies conformal invariance for supersymmetric theories. In that case, the “strongest” version of the  $a$ -theorem might still be valid for supersymmetric theories.

## 2.6. Why Dilatation Generators Generate Dilatations

The authors of Ref. [11] showed that dilatation generators do not generate dilatations (in non-scale-invariant quantum field theories). They demonstrated at low orders that quantum anomalies can be absorbed into a redefinition of the scaling dimensions of the fields, but that at high orders this is not possible. In modern language, the two anomalies<sup>9</sup> correspond to the anomalous dimensions and the beta functions. The former can be safely absorbed into a redefinition of the scaling dimensions of the fields, preserving scale invariance, but the latter, generically cannot, thus breaking scale invariance in the quantum theory. This is made manifest by the trace anomaly, the statement that once the equations of motions are applied the trace of the energy momentum tensor vanishes if and only if the beta-functions vanish.

It is interesting to see how scale-invariant quantum field theories circumvent the results of Ref. [11]. At conformal fixed points the vertex corrections are fixed in terms of the anomalous dimensions, as can be seen by setting the beta functions to zero in Eqs. (2.6). By contrast, for a theory to be scale-invariant, Eqs. (2.8) must be satisfied, in which case the vertex corrections satisfy Eqs. (2.9). Therefore, as in CFTs, the vertex corrections are fixed, only now they are not given in terms of the anomalous dimensions, but rather in terms of the generalized anomalous dimensions,  $\gamma + Q$  for scalars and  $\gamma + P$  for spinors. Consequently, on scale-invariant trajectories it is possible to absorb both the anomalous dimensions *and* the beta functions into a redefinition of the scaling dimensions of the fields, thus preserving scale invariance and leading to dilatation generators that generate dilatations.

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<sup>9</sup>In Ref. [11] a third type of anomalies, the Schwinger terms, arose in the Callan–Symanzik equations for conserved currents. However, Schwinger terms do not arise in dimensional regularization and thus can be safely ignored here.

To see more precisely how this works, recall that the naive Ward identity of scale invariance is improved to the Callan–Symanzik equation [11] in the quantum theory. Indeed, for an arbitrary collection of fields  $\varphi_i(x)$  and couplings  $g_i$ , defined at the renormalization scale  $M$ , the effective action  $\Gamma[\varphi(x), g, M]$  satisfies

$$\left[ M \frac{\partial}{\partial M} + \beta^i \frac{\partial}{\partial g^i} + \gamma_j^i \int d^4x \varphi_i(x) \frac{\delta}{\delta \varphi_j(x)} \right] \Gamma[\varphi(x), g, M] = 0.$$

The authors of Ref. [11] pointed out that the beta-function contributions cannot be absorbed into a redefinition of the dilatation current, and thus generators of dilatations do not generate dilatations, except, of course, if the theory is conformal. However, for scale-invariant theories the beta functions have very specific linear dependence on the couplings. If  $\delta\phi_j = Q_j^i \phi_i$  is an infinitesimal transformation that is a symmetry of the kinetic terms, then a formal symmetry-relation is obtained by counter-rotating the coupling constants, leading to

$$\left[ -Q_j^i g^j \frac{\partial}{\partial g^i} + Q_j^i \int d^4x \varphi_i(x) \frac{\delta}{\delta \varphi_j(x)} \right] \Gamma[\varphi(x), g, M] = 0.$$

Then, using Eqs. (2.8) it is obvious that we can substitute the beta-function terms for new anomalous-dimension-like terms, which lead to a new version of the Callan–Symanzik equation for the effective action:

$$\left[ M \frac{\partial}{\partial M} + (\gamma_j^i + Q_j^i) \int d^4x \varphi_i(x) \frac{\delta}{\delta \varphi_j(x)} \right] \Gamma[\varphi(x), g, M] = 0.$$

We have managed to recast the Callan–Symanzik equation as if the beta functions vanished and the anomalous dimensions were the generalized ones. This is not to say that the couplings do not flow. Rather, the flow is precisely such that one can equivalently solve the equations as if the beta functions vanished but the anomalous dimensions were replaced by the generalized ones. Clearly, there is no longer an obstruction to absorbing the beta-function contributions into a redefinition of the dilatation current.

It is reassuring that we can arrive at the same conclusion by a different, largely independent argument, namely by analyzing the Poincaré algebra augmented by dilatation transformations. The beta functions on scale-invariant trajectories generate the appropriate scaling dimensions required by the inclusion of the virial current in the dilatation current. Classically, it is easy to see that the dilatation current (2.1) leads to new “classical” contributions to the scaling dimensions of the fields by using the Lie algebra of Poincaré and dilatation transformations. The Poincaré algebra, augmented with the dilatation charge,

$D = \int d^3x \mathcal{D}^0(x)$ , is

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho}), \\ [M_{\mu\nu}, P_\rho] &= -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \\ [D, P_\mu] &= -iP_\mu, \end{aligned}$$

and the algebra acts on fields<sup>10</sup>  $\mathcal{O}_I(x)$  as

$$\begin{aligned} [M_{\mu\nu}, \mathcal{O}_I(x)] &= -i(x_\mu\partial_\nu - x_\nu\partial_\mu + \Sigma_{\mu\nu})\mathcal{O}_I(x), \\ [P_\mu, \mathcal{O}_I(x)] &= -i\partial_\mu\mathcal{O}_I(x), \\ [D, \mathcal{O}_I(x)] &= -i(x \cdot \partial + \Delta)\mathcal{O}_I(x), \end{aligned} \tag{2.15}$$

where  $\Sigma_{\mu\nu}$  are the appropriate spin matrices and  $\Delta = \Delta^{\text{cl}} + \gamma$  are the scale-dimension matrices.

Since the dilatation current is given by Eq. (2.1), we find (with the help of Eqs. (2.15))

$$\begin{aligned} [D, \phi_a(x)] &= -i(x \cdot \partial + 1)\phi_a(x) - iQ_{ab}\phi_b(x), \\ [D, \psi_i(x)] &= -i(x \cdot \partial + \tfrac{3}{2})\psi_i(x) - iP_{ij}\psi_j(x), \end{aligned}$$

and thus the virial current leads to new “classical” contributions to the classical scaling-dimension matrices as anticipated above:

$$\begin{aligned} \Delta_{ab}^{\text{cl}} &= \delta_{ab} + Q_{ab}, \\ \Delta_{ij}^{\text{cl}} &= \tfrac{3}{2}\delta_{ij} + P_{ij}. \end{aligned}$$

In the analysis of the RG flow, however, these new “classical” contributions are introduced at the quantum level by the beta functions as dictated by Eqs. (2.8) or (2.9). In other words, although from Eq. (2.1) the virial current contributions to the scaling dimensions seem to have a classical origin, in the renormalization group analysis they are really generated by quantum corrections, i.e., the beta functions. Thus, the scaling dimensions of the fundamental fields on scale-invariant trajectories are

$$\begin{aligned} \Delta_{ab} &= \delta_{ab} + Q_{ab} + \gamma_{ab}, \\ \Delta_{ij} &= \tfrac{3}{2}\delta_{ij} + P_{ij} + \gamma_{ij}. \end{aligned} \tag{2.16}$$

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<sup>10</sup>Note that the notion of quasi-primary fields is vacuous in a scale-invariant theory without conformal invariance, since the generator of special conformal transformations,  $K_\mu$ , does not exist. Descendants, i.e., operators obtained by the action of the generator of translations  $P_\mu$ , are however well-defined.

Note that the “classical” scaling dimensions are constant along the scale-invariant RG trajectory, while the anomalous dimensions, due to Eqs. (2.12), are not. However, the eigenvalues of the scaling dimensions (2.16) are RG-invariant.

We would like to stress here that the beta functions are *not* shifted away into the scale dimensions of the fields. Some might argue that the beta functions on scale-invariant trajectories should naturally be absorbed into the scale dimensions of the fields, leading to scale-invariant fixed points instead of scale-invariant trajectories. But what does the shift physically mean? The only possible way to modify the beta functions without changing the theory is by performing a scheme change. If such a scheme change existed, it would imply that there is a scheme where the new improved energy-momentum tensor is traceless, leading to conformal invariance. However, since physics is scheme-independent, that is clearly impossible. Furthermore, since the shift is global, it affects all theory space, transforming scale-invariant trajectories into scale-invariant “fixed points” *and* conformal fixed points into conformal “trajectories”. However, following Ref. [2] no traceless symmetric energy-momentum tensor exists on these conformal trajectories, contradicting the general result of Ref. [2]. The shift is thus physically ill-defined and it is unwarranted to demand vanishing beta functions on scale-invariant trajectories.

We can also investigate the vacuum structure of the scale-invariant theory from the tree-level potential. By inspection of the Lagrangian (2.5) we immediately conclude that, if the tree-level scalar potential is bounded from below, then the theory on a scale-invariant trajectory has a stable vacuum (i.e., a global minimum) at the origin of field space, with vanishing energy density. If the potential is not bounded from below, then the theory does not have a stable vacuum. The possibility of flat directions is not considered. Due to Eqs. (2.10) and the fact that the matrices (2.11) are orthogonal/unitary, this statement is RG-invariant as expected. The same is true for CP conservation and the location of the vacuum in field space. Notice that an analysis of the effective potential at one loop does not lead to a better understanding of the theory around the origin of field space. Indeed, due to the relative size of the different couplings on scale-invariant trajectories as seen in Eqs. (3.1), the one-loop contribution to the effective potential cannot balance the tree-level contribution. Thus, the apparent new minimum of the one-loop effective potential, located exponentially close to the origin, involves large logarithms and lies outside the regime of validity of the one-loop approximation. Although it is impossible to determine if the origin of field space is a true minimum from the effective potential point of view, the RG analysis strongly suggests that it is.

Finally, it is of interest to study the behavior of correlation functions of the fields. It is

well-known that scale invariance, along with invariance under the Poincaré group, restricts the form of two-point and three-point correlation functions of fields [21].<sup>11</sup> Focusing on scalar fields  $\mathcal{O}_I(x)$  with scaling dimensions  $\Delta_I$  for simplicity, the two-point and three-point correlation functions must be

$$\begin{aligned}\langle \mathcal{O}_I(x_1) \mathcal{O}_J(x_2) \rangle &= \frac{g_{IJ}}{(x_1 - x_2)^{\Delta_I + \Delta_J}}, \\ \langle \mathcal{O}_I(x_1) \mathcal{O}_J(x_2) \mathcal{O}_K(x_3) \rangle &= \sum_{\substack{\delta_1 + \delta_2 + \delta_3 = \\ \Delta_I + \Delta_J + \Delta_K}} \frac{c_{IJK}^{\delta_1 \delta_2 \delta_3}}{(x_1 - x_2)^{\delta_1} (x_2 - x_3)^{\delta_2} (x_3 - x_1)^{\delta_3}},\end{aligned}$$

where  $g_{IJ}$  and  $c_{IJK}^{\delta_1 \delta_2 \delta_3}$  are constant. Note that, contrary to conformal field theories, two-point correlation functions of fields with different dimensions do not necessarily vanish for scale-invariant theories. Three-point correlation functions are even less constrained. Concentrating on fundamental fields, the form of the two-point functions for real scalar fields can be obtained from the algebra:

$$\langle \phi_a(x) \phi_b(0) \rangle = \left[ (x^2)^{-\frac{\Delta}{2}} G^\phi (x^2)^{-\frac{\Delta^T}{2}} \right]_{ab},$$

where  $G_{ab}^\phi$  is a constant real symmetric matrix.

Finally, since the operator product expansion (OPE) already incorporates scale invariance, no new constraints arise for the OPE on scale-invariant trajectories. This is in contrast to conformal theories where very powerful results can be derived with the use of the generator of special conformal transformations on the OPE.

### 3. Scale-invariant Trajectories

#### 3.1. Systematic Approach

As explained in [1] it is possible to systematically search for conformal fixed points (where the virial current trivially vanishes), conformal fixed points with enhanced symmetry (where the virial current is conserved), scale-invariant trajectories (where the virial current is non-trivial), and scale-invariant trajectories with enhanced symmetry (where the non-trivial virial current can be decomposed into a conserved current and a new non-trivial virial current) in the weak-coupling regime. One simply needs to expand in the small parameter

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<sup>11</sup>Although quasi-primary fields are unique to conformal field theories, it is still possible in non-conformal scale-invariant theories to limit the study of correlation functions only to fields that are not descendants.

$\epsilon$  the couplings,

$$g_A = \sum_{n \geq 1} g_A^{(n)} \epsilon^{n-\frac{1}{2}}, \quad \lambda_{abcd} = \sum_{n \geq 1} \lambda_{abcd}^{(n)} \epsilon^n, \quad y_{a|ij} = \sum_{n \geq 1} y_{a|ij}^{(n)} \epsilon^{n-\frac{1}{2}}, \quad (3.1)$$

and the unknown parameters in the virial current,

$$Q_{ab} = \sum_{n \geq 2} Q_{ab}^{(n)} \epsilon^n, \quad P_{ij} = \sum_{n \geq 2} P_{ij}^{(n)} \epsilon^n. \quad (3.2)$$

The form of the expansions (3.1) and (3.2) is dictated by the beta functions for the couplings [22] and by the Polchinski–Dorigoni–Rychkov argument for the virial current [2, 8]. The only requirement on the small parameter is to allow a (partial) cancellation of the first and second non-trivial contributions to the beta functions. For example, the small parameter can be the  $\epsilon$  of  $d = 4 - \epsilon$ , or the specific function of the number of colors and flavors in a theory of the Banks–Zaks type in  $d = 4$  [23].<sup>12</sup> It is natural to ask what happens to the scale-invariant trajectories in the strong-coupling regime. Once the non-perturbative effects are large, all confidence in the expansion above is lost, and it is impossible to argue for the existence of scale-invariant trajectories. However, one can imagine that there are scale-invariant trajectories that have sections both in the perturbative and the non-perturbative regime. It is also likely that the scale-invariant trajectories survive in an intermediate regime as one transitions to strong coupling. For example for a theory in  $d = 4 - \epsilon$  the RG flows in the  $\epsilon \rightarrow 1$  limit may give (strongly-coupled) examples of cycles in  $d = 3$ . Moreover, in a theory of the Banks–Zaks type one could assume that scale-invariant trajectories exist in the strong-coupling regime, much as is done for the conjectured superconformal fixed points in Ref. [25]. However, one cannot parallel the argument of Ref. [25] for the case of scale-invariant trajectories, since it relies on the full superconformal symmetry of the fixed points, more specifically on the existence of an R-symmetry. The same can be said of supersymmetric scale-invariant trajectories if such theories exist. Before concluding, we would like to comment on the scheme-dependence of beta functions.

### 3.2. Scheme-dependence of Beta Functions

It is well known that only the first two terms in the loop expansion of the beta function of QCD are scheme-independent, and hence that a scheme exists for which the beta function consists precisely of those first two terms [26]. The situation for models with multiple

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<sup>12</sup>Notice that, in the large number of colors and flavors limit, it is more natural to use generalized 't Hooft couplings [24] for all couplings in Eqs. (3.1) and (3.2).



couplings is only slightly more complex, but seems to be less well understood. This may well have been discussed elsewhere, but we have tried and failed to find it in the literature. So we will point out here that, for multi-coupling beta functions, although the one-loop terms are scheme-independent, the two- and higher-loop terms are not. However, there is in general no scheme choice that can shift to zero the two- or higher-loop contributions to the beta functions.

We will arrange all the coupling constants of the model into one vector  $g_\alpha = (g_1, \dots, g_N)$ , where the entries stand for squares of Yang–Mills couplings,  $g_A^2$  where  $A$  runs over the group factors of the gauge group, single powers of scalar quartic couplings,  $\lambda_I$  with  $I$  running over all possible quartic operators, and products of Yukawa couplings,  $y_i y_j$  (taking all Yukawa couplings to be real by separating real from imaginary parts) with  $i$  running over the number of Yukawa terms. Then, the loop expansion of the running of all coupling constants can be written in a unified way:

$$\mu \frac{dg_\alpha}{d\mu} \equiv \beta_\alpha = b_{\alpha\beta\gamma}^{(1)} g_\beta g_\gamma + b_{\alpha\beta\gamma\delta}^{(2)} g_\beta g_\gamma g_\delta + \dots$$

Consider, next, a re-parametrization of these coupling constants:

$$\bar{g}_\alpha = g_\alpha + A_{\alpha\beta\gamma} g_\beta g_\gamma + \dots \quad (3.3)$$

This can be seen to correspond to a change in renormalization scheme. Indeed, if the re-parametrization in Eq. (3.3) is introduced into the renormalized Lagrangian, one can simply absorb it into the coupling constant renormalization factors  $Z_g$ . But this has the effect of additional finite subtractions, that is, a change in scheme.

In terms of the new coupling constants the beta functions read

$$\bar{\beta}_\alpha = \bar{b}_{\alpha\beta\gamma}^{(1)} \bar{g}_\beta \bar{g}_\gamma + \bar{b}_{\alpha\beta\gamma\delta}^{(2)} \bar{g}_\beta \bar{g}_\gamma \bar{g}_\delta + \dots$$

It is now a simple exercise to relate the coefficients of the loop expansion of one set of couplings and the other:

$$\bar{b}_{\alpha\beta\gamma}^{(1)} = b_{\alpha\beta\gamma}^{(1)}, \quad (3.4a)$$

$$\bar{b}_{\alpha\beta\gamma\delta}^{(2)} = b_{\alpha\beta\gamma\delta}^{(2)} + \frac{2}{3} [(A_{\alpha\delta\rho} b_{\rho\beta\gamma}^{(1)} - A_{\rho\delta\beta} \bar{b}_{\alpha\rho\gamma}^{(1)}) + \text{permutations}], \quad (3.4b)$$

where the permutations are over all indices but  $\alpha$ . We immediately see that the one-loop terms are scheme-independent, while the two-loop terms are not. Moreover, we also notice that  $A$  does not have enough parameters to allow us to set to zero the two-loop coefficients (3.4b). The same holds for all higher-loop coefficients of the beta functions.

In dimensional regularization the two-loop beta functions, when evaluated on scale-invariant trajectories, are of the order of the three-loop terms, as will be described in future work. Thus, without the knowledge of the three-loop beta functions, one cannot argue consistently that a solution obtained using the two-loop beta functions corresponds to a scale-invariant recurrent flow rather than a fixed point. It should be noted that generally there does not exist a scheme in which the beta functions are two-loop exact. One thus concludes that a three-loop computation is necessary to establish the existence of scale-invariant trajectories. These issues will be investigated in great detail in another publication.

#### 4. Discussion and Conclusion

It has long been thought that scale invariance implies conformal invariance in unitary quantum field theory. A proof in  $d = 2$  spacetime dimensions was known, but no such proof existed for  $d > 2$  spacetime dimensions. In this work we lay down the theoretical foundations for  $d = 4 - \epsilon$  and  $d = 4$  unitary quantum field theories with finite correlation functions that are scale but not conformally invariant.

On scale-invariant trajectories the dilatation current is conserved, the virial current has dimension exactly 3, although it is not conserved (something conformal symmetry does not allow), and the RG evolution is known precisely. Moreover, scale-invariant trajectories exhibit recurrent behaviors (limit cycles or ergodicity) with non-vanishing beta functions. This fact implies that RG flows are not gradient flows and, therefore, the “strongest” version of the  $a$ -theorem is violated. Finally, dilatation generators do generate dilatations on scale-invariant trajectories, since the beta functions can also be absorbed into a redefinition of the scaling dimensions of the fields. Indeed, the beta functions generate the appropriate scaling dimensions required by the non-trivial contribution of the virial current to the dilatation current. As expected, statements such as boundedness of the scalar potential, CP conservation, and location of the minimum in field space are RG-invariant for scale-invariant theories. Note, however, that an analysis of the effective potential around the origin of field space lies outside the range of validity of the one-loop approximation.

Several explicit counterexamples which display scale without conformal invariance will be exhibited elsewhere. Such examples allow the study of the implications of scale invariance, without the added constraint of conformal invariance. They also shatter all hopes for a generic proof that scale implies conformal invariance in arbitrary spacetime dimensions. The perturbative analysis moreover suggests that scale-invariant theories that are

not conformal are generic.

Due to their presumed non-existence, theories with scale but without conformal invariance have been scantily studied. For example, it would be of interest to study the character (attractive, repulsive, etc) of scale-invariant trajectories in generic quantum field theories. Possible phenomenological applications could emerge and result in new ideas for model building. Moreover, an analysis in  $d > 2$  and  $d \neq 4$  spacetime dimensions would also shed light on possible violations of the  $a$ -theorem in other spacetime dimensions. Furthermore, a study of scale without conformal invariance in supersymmetric quantum field theories could also lead to interesting consequences for the  $a$ -theorem. Finally, it would be interesting to investigate scale-invariant theories in the context of gauge/gravity duality [27]. We look forward to addressing some of these questions in the future.

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